Fractals as macroscopic manifestation of squeezed coherent states and brain dynamics

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Abstract. Recent results on the relation between self-similarity and squeezed coherent states are presented. I consider fractals which are generated iteratively according to a prescribed recipe, the so-called deterministic fractals. Fractal properties are incorporated in the framework of the theory of the entire analytical functions and deformed coherent states. Conversely, fractal properties of squeezed coherent states are recognized. This sheds some light on the understanding of the dynamical origin of fractals and their global nature emerging from local deformation processes. The self-similarity in brain background activity suggested by laboratory observations of power-law distributions of power spectral densities of electrocorticograms is also discussed and accounted in the frame of the dissipative many-body model of brain.

1. Introduction
In this paper I will show that a functional representation of self-similarity (as the one occurring in fractals) is provided by squeezed coherent states. Deterministic fractals, which are generated iteratively according to a prescribed recipe, are studied and their self-similarity property, which is considered to be the *most important property* of fractals (p. 150 in [1]), is framed in the theory of the entire analytical functions and coherent states. Conversely, we also recognize, in some specific sense, fractal properties of coherent states. This provides an insight into the dynamical origin of fractals. It also illustrates the geometrical (fractal) properties of coherent states. The conclusion is that fractals are global systems arising from local deformation processes. Therefore they cannot be purely geometric objects. The so called "random fractals", i.e. those fractals obtained by randomization processes introduced in their iterative generation, will not be considered. Since self-similarity is still a characterizing property in many of such random fractals, my conjecture is that also in such cases there must exist a connection with the algebraic structure of coherent states.

The study of the fractal self-similarity property has been also motivated by the laboratory observation of the brain background activity [2]. It results, indeed, that phase patterns observed in neocortical electroencephalograms (EEG) and electrocorticograms (ECoGs) exhibit power-law distributions with no detectable minima. In other words, the neocortical activity appears to be scale-free [2]-[9]. In my presentation I will frame these observations in the context of the dissipative quantum model of brain [10] and non-homogeneous boson condensation in quantum field theory (QFT) [11].

The conjecture that a relation between fractals and the algebra of coherent states exists was originally presented in [12]. As a matter of facts, in several instances fractal properties have been suspected that might be related in someway to coherent structures of the system under study. However, a mathematical proof of such a relation has never been worked out. The formal proof that this relation between fractals and coherent states actually exists and which ones are its specific features have been presented firstly in [13] and [14] (see also [15, 16]), which I will closely follow in the present paper. Certainly, in view of the ubiquitous presence of fractals in nature, the fact that they appear as *macroscopic quantum systems*, namely as macroscopic manifestation of the underlying quantum coherent dynamics, opens a wide horizon of possible applications in physics as well as in biology and in other disciplines, with related boost to interdisciplinary studies. In particular in biology, quantum theories might then introduce new perspectives in the study of basic dynamic phenomena underlying the rich phenomenology of biochemistry, as indeed it has been and it is pursued already in brain modeling and in living matter physics (for an extended discussion and references see [7, 10, 17]).

As we will see in the following discussion, the fact that fractals appear or emerge as macroscopic manifestation of the underlying coherent dynamics is described in a precise and explicit way, so that the word *emergence* acquires a mathematically well defined sense, in contrast to its somewhat vague and even mysterious way it is used often in the literature.

The paper is organized as follows: the formal relation between self-similarity and the coherent states is obtained in Section 2, where the fractal operator is defined. Its relation with the coherent state squeezing generator is presented in Section 3, where some aspects of noncommutative geometry implicit in the fractal generation are also considered. Final remarks and conclusions are presented in Section 4.

2. Self-similarity and the Fock-Bargmann representation of coherent states

Let me start by considering the familiar cases of lengths, surfaces and volumes. For example, a square S whose side is L_0 scales to $(1/2^2)S$ when $L_0 \to \lambda L_0$ with $\lambda = 1/2$. A cube V of same side with same rescaling of L_0 scales to $(1/2^3)V$. Of course, the length L_0 scales to $(1/2)L_0$. Note that $S(\frac{1}{2}L_0)/S(L_0) \equiv p = 1/4$ and $V(\frac{1}{2}L_0)/V(L_0) \equiv p = 1/8$, respectively, so that in both cases $p = \lambda^d$, with $d = 2$ and $d = 3$ for surfaces and volumes, respectively. For the length L_0, it is $p = 1/2 = 1/2^d = \lambda^d$ and of course $d = 1$. In general, by denoting lengths, surfaces or volumes by $\mathcal{H}(L_0)$ the scaling law

$$\mathcal{H}(\lambda L_0) = \lambda^d \mathcal{H}(L_0) \tag{1}$$

holds when lengths are (homogeneously) scaled as $L_0 \to \lambda L_0$.

Assume now that such a scaling law is valid in the case of any other "hypervolume" $\mathcal{H}$. One considers then the ratio

$$\frac{\mathcal{H}(\lambda L_0)}{\mathcal{H}(L_0)} \equiv p \ , \tag{2}$$

and due to Eq. (1) one obtains

$$p \, \mathcal{H}(L_0) = \lambda^d \mathcal{H}(L_0) \ , \tag{3}$$

i.e. $p = \lambda^d$.

Let me apply this to the *Koch curve* (Fig. 1). The discussion can be extended to other iteratively constructed fractals. In the case of the Koch curve one starts by considering a one-dimensional, $d = 1$, segment u_0 of unit length L_0, called the *initiator* [18]. This is called the step, or stage, of order $n = 0$. The length L_0 is then divided by the reducing factor $s = 3$, and the rescaled unit length $L_1 = (1/3)L_0$ is adopted to construct the new "deformed segment" u_1 made of $\alpha = 4$ units L_1 (step of order $n = 1$). u_1 is called the *generator* [18]. Note that such a "deformation" of the u_0 segment is only possible provided that one "gets out" of the
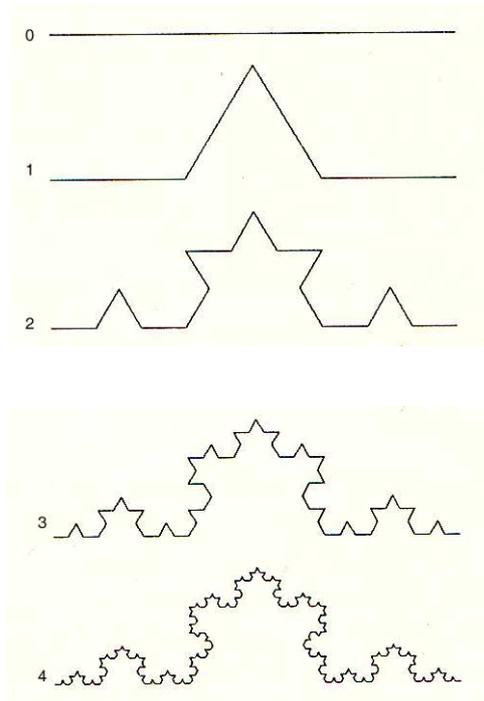
Figure 1. The first five stages of Koch curve.

one-dimensional straight line r to which the u_0 segment belongs: this means that in order to construct the u_1 "shape" the one-dimensional constraint $d = 1$ has to be relaxed: the shape, made of $\alpha = 4$ units L_1, lives in some $d \neq 1$ dimensions. Thus we write

$$u_{1,q}(\alpha) \equiv q \, \alpha \, u_0, \qquad q = \frac{1}{3^d}, \qquad d \neq 1, \tag{4}$$

where d has to be determined. Setting $\alpha \equiv 1/p = 4$ and $q \equiv \lambda^d = 1/3^d$, $p = \lambda^d$ gives

$$q\alpha = 1 \, , \quad \text{where} \quad \alpha = 4, \quad q = \frac{1}{3^d} \, . \tag{5}$$

The meaning of Eq. (4) is that in the "deformed space", to which $u_{1,q}$ belongs, the set of four segments of which $u_{1,q}$ is made "equals" (is equivalent to) the three segments of which u_0 is made in the original "undeformed space". In other words, we require that the measure of the deformed segment $u_{1,q}$ with respect to the undeformed segment u_0 be 1: $u_{1,q}/u_0 = 1$, namely $\alpha q = 4/3^d = 1$.

Steps of higher order n, $n = 2, 3, 4, ..\infty$, can be obtained by iteration of the deformation process keeping $q = 1/3^d$ and $\alpha = 4$. For the nth order deformation we have

$$u_{n,q}(\alpha) \equiv (q \, \alpha) \, u_{n-1,q}(\alpha) \, , \quad n = 1, 2, 3, ... \tag{6}$$

i.e., for any $n \in \mathcal{N}_+$

$$u_{n,q}(\alpha) = (q \, \alpha)^n \, u_0 \, . \tag{7}$$

By proceeding by iteration, or, equivalently, by requiring that $u_{n,q}(\alpha)/u_0$ be 1 for any n, gives $(q \, \alpha)^n = 1$ (in the following, for brevity I will thus set $u_0 = 1$, whenever no misunderstanding

arises). From this and Eq. (5) we obtain, for any n,

$$d = \frac{\ln 4}{\ln 3} \approx 1.2619 \ , \tag{8}$$

which is called the *fractal dimension*, or the *self-similarity dimension* [1]. The (fractal) dimension d is the solution of the relation $1/\alpha = 1/4 = 1/3^d = q$, which for $d = 1$ would be trivially wrong. d is a measure of the "deformation" of the $u_{n,q}$-space *with respect to the u_0-space*. It is important to stress that the fractal is mathematically defined in the limit $n \to \infty$ of the deformation process. The fractal dimension is indeed defined by starting from $(q\alpha)^n = 1$ in the $n \to \infty$ limit. Also notice that since $L_n \to 0$ for $n \to \infty$, the Koch fractal is a curve which is everywhere non-differentiable [1].

The *self-similarity* property of a large class of fractals (the Sierpinski gasket and carpet, the Cantor set, etc.) is expressed in analytical form by Eqs. (6) and (7): "cutting a piece of a fractal and magnifying it isotropically to the size of the original, both the original and the magnification look the same" [18]. In this sense fractals are "scale free", namely viewing a picture of part of a fractal one cannot deduce its actual size if the unit of measure is not given in the same picture [19]. Note that self-similarity does not hold when considering only a finite number n of iterations. Self-similarity is defined only in the $n \to \infty$ limit.

I also observe that the deformation of u_0 into $u_{1,q}$ has been done by varying from 3 to 4 the number α of rescaled unit segments. α and its derivative $\frac{d}{d\alpha}$ are thus expected to play a role in the fractal structure. $(\alpha, \frac{d}{d\alpha})$ play indeed the role of conjugate variables, as we will see (cf. Eq. (11). Invariance, in the limit of $n \to \infty$ iterations, only under anisotropic magnification is called self-affinity. The results here presented can be extended to self-affine fractals. The measure of lengths in fractals, the Hausdorff measure, the fractal "mass" and other fractal properties will not be discussed in this paper.

The main observation is now that Eq. (7) expresses the deep formal connection with the theory of coherent states and the related algebraic structure. Let me make explicit such a connection. Consider in full generality the complex α-plane. The functions

$$u_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}} \ , \qquad u_0(\alpha) = 1 \ , \qquad n \in \mathcal{N}_+ \ , \quad \alpha \in \mathcal{C} \ , \tag{9}$$

form in the vector space $\mathcal{F}$ of the entire analytic functions a basis which is orthonormal under the gaussian measure $d\mu(\alpha) = \frac{1}{\pi} e^{-|\alpha|^2} d\alpha d\bar{\alpha}$. The normalization condition with respect to the gaussian measure is ensured by the factor $\frac{1}{\sqrt{n!}}$, $n \in \mathcal{N}_+$, in Eq. (9). Apart from such a normalization factor, the functions $u_{n,q}(\alpha)|_{q \to 1}$ in Eq. (7) are thus immediately recognized to be nothing but the restriction to real α of the functions in Eq. (9). The study of the fractal properties may be thus carried on in the space $\mathcal{F}$ of the entire analytic functions, by restricting, at the end, to real α, $\alpha \to Re(\alpha)$. Since in Eq. (7) it is $q \neq 1$ ($q < 1$), we also need to consider the "q-deformed" algebraic structure of which the space $\mathcal{F}$ provides a representation.

Actually the space $\mathcal{F}$ provides the Fock-Bargmann representation (FBR) [20, 21] of the Weyl–Heisenberg algebra generated by the set of operators $\{a, a^\dagger, \mathbf{1}\}$:

$$[a, a^\dagger] = \mathbf{1} \ , \qquad [N, a^\dagger] = a^\dagger \ , \qquad [N, a] = -a \ , \tag{10}$$

where $N \equiv a^\dagger a$, with the identification:

$$N \to \alpha \frac{d}{d\alpha} \ , \qquad a^\dagger \to \alpha \ , \qquad a \to \frac{d}{d\alpha} \ . \tag{11}$$

The functions (vectors) $u_n(\alpha)$ (Eq. (9)) are eigenkets of N with integer (positive and zero) eigenvalues. The FBR is the Hilbert space $\mathcal{K}$ generated by the $u_n(\alpha)$, i.e. the whole space $\mathcal{F}$ of the entire analytic functions. Any vector $|\psi\rangle$ in $\mathcal{K}$ is associated, in a one-to-one correspondence, with a function $\psi(\alpha) \in \mathcal{F}$ and is thus described by the set $\{c_n;\ c_n \in \mathcal{C},\ \sum_{n=0}^{\infty} |c_n|^2 = 1\}$ defined by its expansion in the complete orthonormal set of eigenkets $\{|n\rangle\}$ of N:

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \quad \rightarrow \quad \psi(\alpha) = \sum_{n=0}^{\infty} c_n u_n(\alpha), \tag{12}$$

$$\langle \psi | \psi \rangle = \sum_{n=0}^{\infty} |c_n|^2 \quad = \quad \int |\psi(\alpha)|^2 d\mu(\alpha) = ||\psi||^2 = 1, \tag{13}$$

$$|n\rangle \quad = \quad \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle\ , \tag{14}$$

where $|0\rangle$ denotes the vacuum vector, $a|0\rangle = 0$, $\langle 0|0\rangle = 1$. The series expressing $\psi(\alpha)$ in Eq. (12) converges uniformly in any compact domain of the α-plane due to the condition $\sum_{n=0}^{\infty} |c_n|^2 = 1$ (cf. Eq. (13)), confirming that $\psi(\alpha)$ is an entire analytic function.

It is well known that the Fock–Bargmann representation provides a simple frame to describe the usual coherent states (CS) [20, 22] $|\alpha\rangle$. One defines the unitary displacement operator $\mathcal{D}(\alpha)$:

$$\mathcal{D}(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a) = \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(\alpha a^\dagger) \exp(-\bar{\alpha}\, a)\ . \tag{15}$$

so that

$$\mathcal{D}^{-1}(\alpha)\, a\, \mathcal{D}(\alpha) = a + \alpha\ . \tag{16}$$

$\mathcal{D}(\alpha)$ is a bounded operator defined on the whole $\mathcal{K}$. It provides a representation of the Weyl–Heisenberg group usually denoted by W_1 [20]. The coherent state $|\alpha\rangle$ is then given by

$$|\alpha\rangle = \mathcal{D}(\alpha)|0\rangle\ , \quad \alpha \in \mathcal{C}\ , \tag{17}$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} u_n(\alpha)|n\rangle, \tag{18}$$

with

$$a|\alpha\rangle = \alpha|\alpha\rangle\ , \quad \alpha \in \mathcal{C}\ . \tag{19}$$

Eq. (18) and (14) show that the state $|\alpha\rangle$ "is made" of the superposition of states $|n\rangle$ of any number n of a-particles (a-quanta or excitations). $|\alpha\rangle$ is said to be a "condensate" of a-particles. The transformation (16) induced by $\mathcal{D}(\alpha)$ on a (or equivalently on the vacuum state $|0\rangle$, Eq. (17)) is thus understood to induce a "condensation" of a-particles in the vacuum $|0\rangle$. Eq. (19) shows that the annihilation of an a-particle in $|\alpha\rangle$ produces a rearrangement of the states in the sum in Eq. (18) such that the state $|\alpha\rangle$ is again reobtained (modulus the overall coefficient α). This expresses the "coherence" of the condensate and the "phase" α (cf. Eq. (15)) is a measure of the strength or degree of coherence, which is thus the result of the phase correlation binding (phase locking) the whole set of modes a condensed in the state $|\alpha\rangle$. We have thus a correlation mediated by a phase (not by a force). Since coherent states are defined only in the limit $n \to \infty$, the summation in Eq. (18) cannot be approximated by any finite number of terms and we thus see that coherent states cannot be obtained by perturbation theory. In quantum field theory one can clearly show that coherence is a non-perturbative phenomenon [11].

The set $\{|\alpha\rangle\}$, for all $\alpha \in \mathcal{C}$, is an overcomplete set of states. In order to extract a complete set of CS from the overcomplete set it is necessary to introduce in the α-complex plane a regular

lattice L, called the von Neumann lattice [20]. For a general discussion on this subject we address the reader to Refs. [20] and [23]. In this last one the von Neumann lattice is discussed also in connection with the deformation of the Weyl-Heisenberg algebra introduced below. Finally, the relation between the CS and the entire analytic function basis $\{u_n(\alpha)\}$ is:

$$u_n(\alpha) = e^{\frac{1}{2}|\alpha|^2} \langle n|\alpha\rangle \ . \tag{20}$$

In order to account for the q-parameter (cf. Eq. (7)) I now introduce the finite difference operator $\mathcal{D}_q$ defined by [24]:

$$\mathcal{D}_q f(\alpha) = \frac{f(q\alpha) - f(\alpha)}{(q-1)\alpha} \ , \qquad f(\alpha) \in \mathcal{F} \ , \quad q = e^\zeta \ , \quad \zeta \in \mathcal{C}. \tag{21}$$

$\mathcal{D}_q$ reduces to the standard derivative for $q \to 1$ ($\zeta \to 0$). In the space $\mathcal{F}$ the commutation relations hold:

$$[\mathcal{D}_q, \alpha] = q^{\alpha \frac{d}{d\alpha}} \quad , \quad \left[\alpha \frac{d}{d\alpha}, \mathcal{D}_q\right] = -\mathcal{D}_q \quad , \quad \left[\alpha \frac{d}{d\alpha}, \alpha\right] = \alpha \ , \tag{22}$$

which, as for Eq. (11), lead us to the identification

$$N \to \alpha \frac{d}{d\alpha} \quad , \quad \hat{a}_q \to \alpha \quad , \quad a_q \to \mathcal{D}_q \ , \tag{23}$$

with $\hat{a}_q = \hat{a}_{q=1} = a^\dagger$ and $\lim_{q\to 1} a_q = a$ on $\mathcal{F}$. The algebra (22) is the q-deformation of the algebra (10). More details on the properties of $\mathcal{D}_q$ and the q-deformed algebra (22) can be found in Refs. [23]. I only recall that the operator q^N acts on the whole $\mathcal{F}$ as

$$q^N f(\alpha) = f(q\alpha) \ , \quad f(\alpha) \in \mathcal{F} \ , \tag{24}$$

which, when applied to the coherent state functional (18), gives

$$q^N |\alpha\rangle = |q\alpha\rangle = \exp\left(-\frac{|q\alpha|^2}{2}\right) \sum_{n=0}^\infty \frac{(q\alpha)^n}{\sqrt{n!}} \, |n\rangle \ . \tag{25}$$

Since $q\alpha \in \mathcal{C}$, from Eq. (17),

$$a|q\alpha\rangle = q\alpha|q\alpha\rangle \ , \quad q\alpha \in \mathcal{C} \ . \tag{26}$$

We thus see that the nth fractal iteration (cf. Eq. (7) where we set $u_0 \equiv 1$) is obtained by projecting out the nth component of $|q\alpha\rangle$ and restricting to real $q\alpha$,

$$u_{n,q}(\alpha) = (q\alpha)^n = \sqrt{n!} \, \exp\left(\frac{|q\alpha|^2}{2}\right) \langle n|q\alpha\rangle, \quad q\alpha \to Re(q\alpha) \ , \quad \text{for any} \ n \in \mathcal{N}_+ \ , \tag{27}$$

which, by using $\langle n| = \langle 0| \frac{(a)^n}{\sqrt{n!}}$, gives

$$u_{n,q}(\alpha) = (q\alpha)^n = \exp\left(\frac{|q\alpha|^2}{2}\right) \langle 0|(a)^n|q\alpha\rangle, \quad q\alpha \to Re(q\alpha), \quad n \in \mathcal{N}_+, \tag{28}$$

This shows that the operator $(a)^n$ acts as a "magnifying" lens [18]: the nth iteration of the fractal can be "seen" by applying $(a)^n$ to $|q\alpha\rangle$ and restricting to real $q\alpha$:

$$\langle q\alpha|(a)^n|q\alpha\rangle = (q\alpha)^n = u_{n,q}(\alpha), \quad q\alpha \to Re(q\alpha). \tag{29}$$

Let me close this Section by observing that Eq. (26) expresses the invariance of the coherent state (representing the fractal) under the action of the operator $\frac{1}{q\alpha}a$. Such an invariance property allows us to consider the coherent functional $\psi(q\alpha)$ as an "attractor" in $\mathcal{C}$. In this sense, Eq. (26) reminds us of the fixed point equation $W(A) = A$, where W is the Hutchinson operator [18], characterizing the iteration process for the fractal A in the $n \to \infty$ limit.

In summary, the operator q^N applied to $|\alpha\rangle$ "produces" the fractal in the functional form of the coherent state $|q\alpha\rangle$ (cf Eq. (25)). The nth fractal stage of iteration, $n = 0, 1, 2, .., \infty$ is represented, in a one-to-one correspondence, by the nth term in the coherent state series in Eq. (25). I call q^N *the fractal operator*.

Eqs. (27), (28) and (29) formally establish the searched connection between fractal self-similarity and the (q-deformed) algebra of the coherent states.

3. The fractal operator and the squeezing generator

In this Section I consider the relation between the fractal operator q^N and the squeezing transformation for coherent states. In the Hilbert space identified with the space $\mathcal{F}$ of the entire analytic functions $\psi(\alpha)$ the following identity holds

$$2\alpha \frac{d}{d\alpha}\psi(\alpha) = \left\{ \frac{1}{2}\left[\left(\alpha + \frac{d}{d\alpha}\right)^2 - \left(\alpha - \frac{d}{d\alpha}\right)^2 \right] - 1 \right\}\psi(\alpha) , \qquad (30)$$

and I will put

$$c = \frac{1}{\sqrt{2}}(\alpha + \frac{d}{d\alpha}) \quad , \quad c^\dagger = \frac{1}{\sqrt{2}}(\alpha - \frac{d}{d\alpha}) \quad , \quad [c, c^\dagger] = \mathbf{1} . \qquad (31)$$

Their relation with the FBR operators a and $a^\dagger$, previously introduced in Eq. (11), is

$$\alpha = \frac{1}{\sqrt{2}}(c + c^\dagger) \ \to a^\dagger , \quad \frac{d}{d\alpha} = \frac{1}{\sqrt{2}}(c - c^\dagger) \ \to a . \qquad (32)$$

In $\mathcal{F}$, $c^\dagger$ is the conjugate of c [20, 23]. By setting $\alpha \equiv x + iy$, and considering the holomorphy conditions holding for $f(\alpha) \in \mathcal{F}$

$$\frac{d}{d\alpha}f(\alpha) = \frac{d}{dx}f(\alpha) = -i\frac{d}{dy}f(\alpha) , \qquad (33)$$

in the $y \to 0$ limit we get form (31)

$$c \to \frac{1}{\sqrt{2}}(x + ip_x) \equiv \hat{z} , \quad c^\dagger \to \frac{1}{\sqrt{2}}(x - ip_x) \equiv \hat{z}^\dagger , \quad [\hat{z}, \hat{z}^\dagger] = 1 , \qquad (34)$$

where $p_x = -i\frac{d}{dx}$. $\hat{z}^\dagger \leftrightarrow a^\dagger$ and $\hat{z} \leftrightarrow a$ are the usual creation and annihilation operators in the configuration representation. Thus we see that in the limit $\alpha \to Re(\alpha)$, i.e. $y \to 0$, c and $c^\dagger$ turn into the conventional annihilation and creator operators, respectively, associated with x and p_x in the canonical configuration representation.

I now observe that the fractal operator q^N can be realized in $\mathcal{F}$ as:

$$q^N \psi(\alpha) = \frac{1}{\sqrt{q}} \ \exp\left(\frac{\zeta}{2}(c^2 - c^{\dagger 2})\right)\psi(\alpha) \equiv \frac{1}{\sqrt{q}}\hat{\mathcal{S}}(\zeta)\psi(\alpha) \equiv \frac{1}{\sqrt{q}}\psi_s(\alpha) , \qquad (35)$$

where $q = e^\zeta$ (for simplicity, assumed to be real) and as usual $N = \alpha \, d/d\alpha$. $\hat{\mathcal{S}}(\zeta)$ is the squeezing generator well known in quantum optics [23, 25]. $\zeta = \ln q$ is called the squeezing parameter and $\psi_s(\alpha)$ denotes the squeezed states in the FBR. From Eq. (35) we see that q^N acts in $\mathcal{F}$, as well

as in the configuration representation in the limit $y \to 0$, as the squeezing operator $\hat{\mathcal{S}}(\zeta)$, up to the numerical factor $1/\sqrt{q}$.

Since $q^N \psi(\alpha) = \psi(q\alpha)$ (cf. Eq. (24)), from Eq. (35) we conclude that the q-deformation process, which we have seen is associated to the fractal generation process, is equivalent to the squeezing transformation.

The squeezing generator $\hat{\mathcal{S}}(\zeta)$ is an $SU(1,1)$ group element. In fact, the operators $K_{-} = (1/2)c^2$, $K_{+} = (1/2)c^{\dagger 2}$, $K_{\alpha} = (1/2)(c^{\dagger}c + 1/2)$, close the algebra $su(1,1)$. We thus may write the $SU(1,1)$ Bogoliubov (squeezing) transformations for the c's operators:

$$\hat{\mathcal{S}}^{-1}(\zeta) \, c \, \hat{\mathcal{S}}(\zeta) = c \, \cosh\zeta - c^{\dagger} \, \sinh\zeta \, , \tag{36}$$

$$\hat{\mathcal{S}}^{-1}(\zeta) \, c^{\dagger} \, \hat{\mathcal{S}}(\zeta) = c^{\dagger} \, \cosh\zeta - c \, \sinh\zeta \, , \tag{37}$$

and in the $y \to 0$ limit

$$\hat{\mathcal{S}}^{-1}(\zeta) \, \alpha \, \hat{\mathcal{S}}(\zeta) \;=\; \frac{1}{q}\alpha \to \frac{1}{q}x \, , \tag{38}$$

$$\hat{\mathcal{S}}^{-1}(\zeta) \, p_{\alpha} \, \hat{\mathcal{S}}(\zeta) \;=\; qp_{\alpha} \to qp_x \, , \tag{39}$$

where $p_{\alpha} \equiv -i\frac{d}{d\alpha}$, and

$$\int d\mu(\alpha)\bar{\psi}(\alpha)\hat{\mathcal{S}}^{-1}(\zeta) \, \alpha \, \hat{\mathcal{S}}(\zeta)\psi(\alpha) \;\to\; \frac{1}{q} <x> \tag{40}$$

$$\int d\mu(\alpha)\bar{\psi}(\alpha)\hat{\mathcal{S}}^{-1}(\zeta) \, p_{\alpha} \, \hat{\mathcal{S}}(\zeta)\psi(\alpha) \;\to\; q <p_x> \tag{41}$$

so that the root mean square deviations Δx and Δp_x satisfy

$$\Delta x \Delta p_x = \frac{1}{2} \quad , \quad \Delta x = \frac{1}{q}\sqrt{\frac{1}{2}} \quad , \quad \Delta p_x = q\sqrt{\frac{1}{2}} \, , \tag{42}$$

which confirms that the q-deformation plays the role of squeezing transformation. Note that the action variable $\int p_x \, dx$ is invariant under the squeezing transformation.

Eq. (38) shows that $\alpha \to (1/q)\alpha$ under squeezing transformation, which, in view of the fact that $q^{-1} = \alpha$ (cf. Eq. (5)), means that $\alpha \to \alpha^2$, i.e. under squeezing we proceed further in the fractal iteration process. In conclusion, the fractal iteration process can be described in terms of the coherent state squeezing transformation.

Use of (38) and (39) leads to

$$\hat{z}_q = \frac{1}{q\sqrt{2}}(x + iq^2 p_x) \, , \quad \hat{z}_q^{\dagger} = \frac{1}{q\sqrt{2}}(x - iq^2 p_x), \quad [\hat{z}_q, \hat{z}_q^{\dagger}] = 1 \, . \tag{43}$$

Notice that the $\hat{z}$ and $\hat{z}^{\dagger}$ algebra is preserved under the squeezing transformation, which is indeed a canonical transformation. In the "deformed" phase space let us denote the coordinates $(x, q^2 p_x)$ as (x_1, x_2), where $x_1 \equiv x$ and $x_2 \equiv q^2 p_x$. Coordinates do not commute in this (deformed) space:

$$[x_1, x_2] = iq^2 \, . \tag{44}$$

We thus recognize that q-deformation introduces non-commutative geometry in the (x_1, x_2)-space. In such a space the noncommutative Pythagora's theorem gives the distance D:

$$D^2 = x_1^2 + x_2^2 = 2q^2(\hat{z}_q^{\dagger}\hat{z}_q + \frac{1}{2}) \, . \tag{45}$$

In $\mathcal{F}$, in the $y \to 0$ limit, from the known properties of creation and annihilation operators we then get

$$D_n^2 = 2q^2(n + \frac{1}{2}) \ , \quad n = 0, 1, 2, 3... \ , \tag{46}$$

i.e. in the space (x_1, x_2) associated to the coherent state fractal representation, the (x_1, x_2)-distance is quantized according to the unit scale set by q. Eq. (46) also shows that in the space (x_1, x_2) we have quantized "disks" of squared radius vector D_n^2. It is interesting to observe that the "smallest" of such disks has non-zero radius given by the deformation parameter q (recall that $q = \frac{1}{3^d}$ when Koch fractal is considered). We also realize that Eq. (46) is nothing but the expression of the energy spectrum of the harmonic oscillator, i.e. one could write Eq. (46) as $D_n^2 = 2q^2(n + \frac{1}{2}) \equiv E_n, \quad n = 0, 1, 2, 3...,$ where E_n might be thought as the "energy" associated to the nth fractal stage.

4. Scale-free laws and brain waves

In laboratory observations the mesoscopic neural activity of neocortex appears to consist of the dynamical formation of spatially extended neuronal domains in which widespread cooperation supports brief epochs of patterned synchronized oscillations [26]. In QFT information is stored in ordered patterns maintained by long range correlations mediated by massless quanta, the so-called Nambu-Goldstone (NG) modes or particles. Memory could be then described as a printed pattern of order supported by long range correlations. In 1967 Umezawa proposed indeed a model of brain dynamics based on the dynamical generation of such long range correlation in the way they are normally treated in many-body theory [27]. Already in the early 1940s Lashley remarked, with reference to the diffuse, non-local nature of brain activity which he observed in laboratory: "...Here is the dilemma. Nerve impulses are transmitted ...from cell to cell through definite intercellular connections. Yet, all behavior seems to be determined by masses of excitation...within general fields of activity, without regard to particular nerve cells... What sort of nervous organization might be capable of responding to a pattern of excitation without limited specialized path of conduction? The problem is almost universal in the activity of the nervous system" (pp. 302–306 of Ref. [28]). Pribram, soon after the discovery of the laser light in the early 1960s [29, 22], proposed his holographic hypothesis [30]. In Umezawa's many-body model for the brain two main ingredients appear: the notion of "field" introduced by Lashley in his puzzling dilemma and the notion of "coherence", intrinsic to the laser theory inspiring Pribram view. Both these notions are basic ones in the QFT dynamics generating ordered patterns.

With the advent of modern advanced technologies, the amplitude modulated (AM) patterns and the phase modulated (PM) patterns in the brain are of common observational access. Laboratory observations show that the observed cortical collective activity cannot be fully accounted for by the electric field of the extracellular dendritic current or the extracellular magnetic field from the high-density electric current inside the dendritic shafts, which are much too weak, or by the chemical diffusion, which is much too slow [7, 8, 31]. Patterns of phase-locked oscillations are intermittently present in resting, awake subjects as well as in the same subject actively engaged in cognitive tasks requiring interaction with environment, so they are best described as properties of the background activity of brains that is modulated upon engagement with the surround. These "packets of waves" extend over spatial domains covering much of the hemisphere in rabbits and cats [3, 32, 33, 34], and over regions of linear size of about 19 cm [35] in human cortex with near zero phase dispersion [6, 36, 37].

Umezawa's many-body model [27] and its extension to dissipative dynamics, the dissipative many-body model of brain [38], are based on the QFT notion of spontaneous breakdown of symmetry (SBS). When the symmetry of the Lagrangian is not the symmetry of the least energy state (the ground state or the vacuum) of the system, the symmetry is said to be spontaneously

broken and mathematical consistency requires the existence of NG massless particles [11, 39]. Examples are the phonon modes in the crystals, the magnon modes in ferromagnets, etc. They can be described, as customary in a quantum theory, as the quanta associated to certain waves, such as the elastic wave in crystals, the spin wave in ferromagnets. The role of such waves is the one of establishing long range correlation among the system constituents. For example, in magnets, the elementary magnetic dipoles are forced to oscillate "in phase" under the correlation established by the spin waves, i.e. by the magnon quanta spanning the extended domain which thus gets characterized by its macroscopic magnetization. The mechanism of spontaneous breakdown of symmetry is at the origin of the *change of scale*: from the microscopic scale of the elementary constituent dynamics to the macroscopic scale of the system magnetization, which is therefore a measure of the ordering of the elementary constituents and is for that reason called the "order parameter". The external field (the trigger or stimulus) is responsible of the "phase transition" from the normal (zero magnetization) phase to the magnetic phase. Then the mathematical structure of the theory must be adequate to allow physically distinct phases (technically called unitarily inequivalent representations of the quantum algebra). QFT possesses indeed such a mathematical structure. The ground states corresponding to physically distinct phases are characterized by distinct degrees of ordering which are described by different densities of NG modes *condensed* in them. Such a condensation of NG modes in the ground states is a *coherent* condensation, which physically expresses the "in phase", i.e. synchronized, elementary constituent oscillations. In quantum mechanics (QM), on the contrary, all the state representations are physically equivalent (unitarily equivalent) and therefore QM is not useful to describe phase transitions [40, 41].

I want to stress that in Umezawa's model and in the dissipative many-body model the neuron and the glia cells are treated as classically behaving systems. The quantum variables relevant to the many-body model of brain have been identified [38, 42] to be the electric dipole vibrational modes of the water molecules which constitutes the matrix in which neurons and glia cells and other mesoscopic units are embedded. The spontaneous breakdown of the rotational symmetry of electrical dipoles of water and other molecules implies the existence of NG modes. In such a context, these NG modes have been called the dipole wave quanta (DWQ) [38, 42, 43]. The system ground state thus is obtained in terms of coherent condensation of the DWQ, which are the agents responsible of the neuronal coordination. In particular it is found that the "memory state" is a squeezed coherent state for the basic quantum variables, whose mesoscopic order parameter expresses, at the synaptic level, the amplitude and phase modulation of the carrier signal observed in EEGs and ECoEGs.

The original Umezawa's brain model suffers of a very much limited memory capacity. Any subsequent stimulus cancels the one produced by the preceding stimulus (overprinting). It has been shown [38] that such a problem may find a solution when the model is modified so to include the crucial property of brain system of being an open system. In this paper I will not insist further in the description of the dissipative many-body model, which can be found in the existing literature [7, 8, 38], [44] - [47]. I will observe only that the dissipative many-body model describes the scale-free and self-similarity property of the brain dynamics. As said, laboratory observations show that self-similarity characterizes the brain ground state. Measurements of the durations, recurrence intervals and diameters of neocortical EEG phase patterns have power-law distributions with no detectable minima. The power spectral densities in both time and space of ECoGs from surface arrays conform to power-law distributions [32, 33, 48, 49, 50]. This suggests that the activity patterns generated by neocortical neuropil might be scale-free [2]-[9], thus explaining the self-similarity in ECoGs patterns over distances ranging from hypercolumns to an entire cerebral hemisphere. On the other hand, one of the main actors in the dissipative model is the (brain) ground state, which turns out to be a squeezed coherent state like the ones described in the previous Sections as *q*-deformed coherent state [38]. Since a functional representation of

self-similarity is provided by squeezed coherent states, we see that the dissipative model of brain accounts for the self-similarity in brain background activity.

5. Conclusion

Let me close the paper with a couple of comments. Certainly, connecting fractal self-similarity and coherent states opens the way to a series of possible theoretical developments and practical applications in a number of research sectors, including the field of complex dynamical systems.

From the theoretical point of view, embedding the fractal study in the framework of the entire analytical functions may contribute to the treatment of fractal properties which present non-trivial difficulties due to the lack of a tractable mathematical representation; for example, the Koch curve is known to be everywhere non-differentiable [1], which makes it greatly interesting, but also difficult to deal with. From the standpoint of coherent states, there might be interesting applications in laser physics and quantum optics, where Glauber coherent states play a dominant role. Since coherent states always emerge as a result of boson condensation (see, e.g., Eq. (16) and the subsequent remarks), recognizing their fractal structure may contribute to the understanding of physical properties and behaviors in condensed matter systems endowed with a condensed ground state. Here, of course, the perspective opens towards the large class of phenomena based on the mechanism of spontaneous breakdown of symmetry. There the physical meaning of the process of coherent condensation of Nambu-Goldstone modes is the one of the appearance of long range correlation in the system, namely of the dynamical formation of ordered patterns in the ground state. The common feature underlying fractal formation and ordered pattern formation is thus in the dynamical emergence of long range correlation modes. This is certainly the case in quantum field theory systems characterized by ordered patterns in the ground state. The discussion presented above seems to suggest that this might be also the case for fractals. If so, this would provide an interesting example of interplay between the mesoscopic/macroscopic dimension of fractals and the microscopic dynamics out of which they are formed. It also provides an interesting example of a relation of global features (long range correlation at mesoscopic/macroscopic level) with local deformation processes (i.e. dynamical properties at microscopic level). On such a kind of relation is based our understanding of the formation of topologically non-trivial defects (also called extended objects), such as vortices, domain walls, boundary defects, and other soliton-like defects in condensed matter physics and high energy physics [13, 52]. These extended objects behave as macroscopic classical objects, but are generated by the microscopic (quantum) dynamics. On the basis of our discussion, now one might propose that perhaps the class of these extended objects might be enlarged so to include also fractals.

The relation between fractal self-similarity and squeezing is also interesting since it is known that squeezed coherent states are also related with dissipative systems [23, 53, 54]. Thus a link might be established between dissipation, which plays a relevant role in complex dynamical systems, and fractals. An interesting result is, for example, the one provided by the measurement [55] of the ph as a function of conductivity of samples of water in contact with hydrophilic films of substances such as the so called nafion. Considering that the ph is the $Log_{10}[H^+]$, where $[H^+]$ denotes the proton concentration in water, the plot ph-$Log_{10}\chi$, with χ denoting the conductivity, is a log-log plot and it is found [55] to be a straight line (with negative angular coefficient). This signals self-similarity and scale-free dynamics. Thus, according to the discussion here presented, it signals the presence of coherent structures in water, formed as a result of the water molecule interaction with the nafion film.

Of course, a large field of applications might be offered by biological systems. An interesting example in such a direction is provided by brain studies [2]-[9], [14]. According to the result of the present paper, deformed coherent states provide the functional representation of self-similarity observed in neuro-phenomenological data (for a discussion on the relation of fractals

with brain/mind, although from a perspective different from the one presented in this paper, see also Ref. [56]).

I recall that the Weyl-Heisenberg representations are labeled by the q-parameter by means of the squeezing transformations [54]. In the infinite volume limit (infinite degrees of freedom) representations labeled by different values of the q-deformation parameter are unitarily inequivalent representations [23]. Changes of the value of the q-parameter induce transitions through unitarily inequivalent representations. The trajectories induced over the space of the representations are shown to be, under quite general conditions, chaotic trajectories [57]. Besides the scale parameter, one might consider also phase parameters and translation parameters (which characterize generalized coherent states, such as $SU(2)$, $SU(1,1)$, etc. coherent states) and relate them to the deformation q-parameter. By changing these parameters in a *deterministic iterated function process*, also called *multiple reproduction copy machine* process [1], the Koch curve may be then transformed into another fractal, e.g. into Barnsley's fern [1]. In the framework presented in this paper, these fractals are then described by corresponding unitarily inequivalent representations in the limit of infinitely many degrees of freedom (infinite volume limit) [13]. In this way one might recover the richness of the variety of "different" fractal shapes obtainable by changing the parameters of the fractal one starts with [1].

In conclusion, by limiting my discussion to deterministic fractals, I have presented the functional realization of fractal self-similarity in terms of the q-deformed algebra of coherent states. Fractal study can be thus incorporated in the theory of the entire analytical functions. From the discussion it appears that the reverse is also true: under convenient choice of the q-deformation parameter and by a suitable restriction to real α, coherent states exhibit fractal properties in the q-deformed space of the entire analytical functions.

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